# ISyE/Math/CS/Stat 525
# Linear Optimization

## 2. The geometry of linear programming

Prof. Carla Michini
University of Wisconsin-Madison

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# Outline

Sec. 2.1 We define a polyhedron as a set described by a finite number of linear equality and inequality constraints.

Sec. 2.2 We study the basic geometric properties of polyhedra in some detail, with emphasis on their "corner points".

Sec. 2.3 We focus on the case where the feasible set is in the standard form $\{x \mid Ax = b, x \geq 0\}$.

Sec. 2.4 We study what happens when corner points arise in a degenerate manner.

Sec. 2.5 We see under which conditions a nonempty polyhedron has corner points.

Sec. 2.6 We see that in this case, the search for optimal solutions to linear programming problems can be restricted to corner points.

Sec. 2.7 We provide an alternative representation of polyhedra.

Sec. 2.8 We present the oldest method for solving linear programming problems.

# General Optimization Problem

Consider the optimization problem

$$\begin{aligned} \text{minimize} \quad & f(x) \\ \text{subject to} \quad & x \in S, \end{aligned}$$

where $f : \mathbb{R}^n \to \mathbb{R}$ and $S \subseteq \mathbb{R}^n$.

**Definition**

The problem is infeasible if $S = \emptyset$.

**Definition**

The problem is unbounded if for each real number $K$ there exists $x \in S$ such that $f(x) < K$.

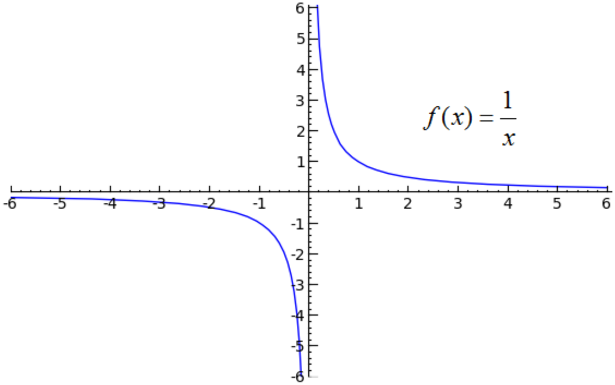
**Definition**

The problem has an optimal solution if there exists $x^* \in S$ such that $f(x^*) \leq f(x)$ for all $x \in S$.

# General Optimization Problem

- In general, an optimization problem can be neither infeasible, nor unbounded, nor have an optimal solution.
- Example:

$$\begin{aligned} \text{minimize} \quad & 1/x \\ \text{subject to} \quad & x \geq 1. \end{aligned}$$



# Linear Programming Problem

Consider the linear programming problem

$$\begin{aligned} \text{minimize} \quad & c'x \\ \text{subject to} \quad & Ax \geq b, \end{aligned}$$

where $c \in \mathbb{R}^n$, $b \in \mathbb{R}^m$ and $A \in \mathbb{R}^{m \times n}$.

We will prove the following fundamental property:

> Property 1:
> Any linear programming problem is either infeasible, or it is unbounded, or it has an optimal solution.

# Local vs Global optimal solutions

Consider the optimization problem

$$\begin{aligned} \text{minimize} \quad & f(x) \\ \text{subject to} \quad & x \in S, \end{aligned}$$

where $f : \mathbb{R}^n \to \mathbb{R}$ and $S \subseteq \mathbb{R}^n$.

**Definition**

A vector $x^* \in S$ is called a <u>(global) optimal solution</u> if

$$f(x^*) \leq f(x) \text{ for all } x \in S.$$

**Definition**

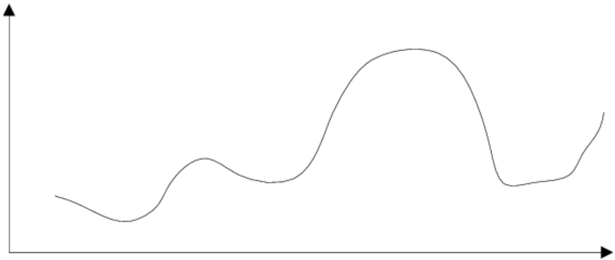
A vector $x^* \in S$ is called a <u>local optimal solution</u> if there exist $\epsilon > 0$ and a neighborhood $N(x^*, \epsilon) = \{x \in \mathbb{R}^n : \|x^* - x\| \leq \epsilon\}$ of $x^*$ such that $f(x^*) \leq f(x)$ for all $x \in S \cap N(x^*, \epsilon)$.

# Local vs Global optimal solutions

Consider the optimization problem

$$\begin{aligned} \text{minimize} \quad & f(x) \\ \text{subject to} \quad & x \in S, \end{aligned}$$

where $f : \mathbb{R}^n \to \mathbb{R}$ and $S \subseteq \mathbb{R}^n$.



There can be local optimal solutions that are NOT global optimal solutions.

# Local vs Global optimal solutions

Consider the linear programming problem

$$\begin{aligned} \text{minimize} \quad & c'x \\ \text{subject to} \quad & Ax \geq b, \end{aligned}$$

where $c \in \mathbb{R}^n$, $b \in \mathbb{R}^m$ and $A \in \mathbb{R}^{m \times n}$.

We will prove the following fundamental property:

> Property 2:
> In a linear programming problem each local optimal solution is also a global optimal solution.

# 2.1 Polyhedra and convex sets

# Hyperplanes, halfspaces, and polyhedra

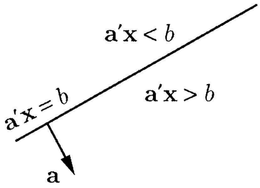
# Hyperplanes and halfspaces

**Definition 2.3**

Let $a$ be a nonzero vector in $\mathbb{R}^n$ and let $b$ be a scalar.

(a) The set $\{x \in \mathbb{R}^n \mid a'x = b\}$ is called a hyperplane.

(b) The set $\{x \in \mathbb{R}^n \mid a'x \geq b\}$ is called a halfspace.




- A hyperplane is the boundary of a corresponding halfspace.
- A hyperplane is equal to the intersection of two halfspaces.
- The vector $a$ in the definition of the hyperplane is perpendicular to the hyperplane itself. Exercise: prove it!

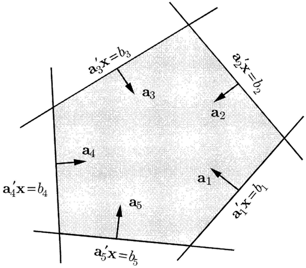
# Polyhedra

**Definition 2.1**

A <u>polyhedron</u> is a set that can be described in the form

$$\{x \in \mathbb{R}^n \mid Ax \geq b\},$$

where $A$ is an $m \times n$ matrix and $b$ is a vector in $\mathbb{R}^m$.




- A polyhedron is the intersection of a finite number of halfspaces.

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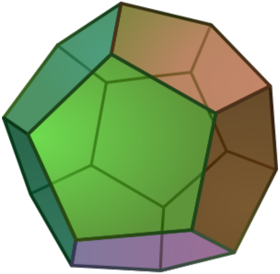


- A polyhedron is the intersection of a finite number of halfspaces.

# Polyhedra

- As discussed in Section 1.1, the feasible set of any linear programming problem can be described by inequality constraints of the form $Ax \geq b$, and is therefore a polyhedron.
- A set of the form

$$\{x \in \mathbb{R}^n \mid Ax = b,\ x \geq 0\}$$

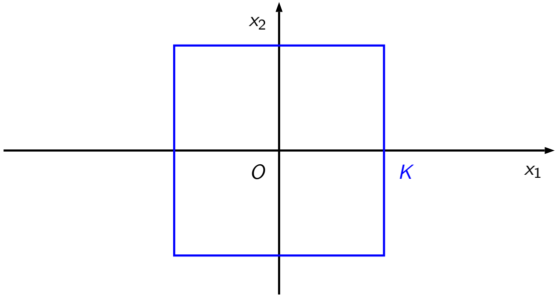
is also a polyhedron, in a standard form representation.

# Bounded Sets and Polytopes

A set can either "extend to infinity" or be confined in a finite region.

**Definition 2.2**

A set $S \subset \mathbb{R}^n$ is <u>bounded</u> if there exists a constant $K$ such that the absolute value of every component of every element of $S$ is less than or equal to $K$.



# Bounded Sets and Polytopes

A set can either "extend to infinity" or be confined in a finite region.

**Definition 2.2**
A set $S \subset \mathbb{R}^n$ is bounded if there exists a constant $K$ such that the absolute value of every component of every element of $S$ is less than or equal to $K$.

**Definition**
A bounded polyhedron is called a polytope.

# Why is Linear Programming special?

[spoilers ahead...]

Property 1:
Any linear programming problem is either infeasible, or it is unbounded, or it has an optimal solution.

Property 2:
In a linear programming problem each local optimal solution is also a global optimal solution.

# Why is Linear Programming special?

[spoilers ahead...]

Property 1:
Any linear programming problem is either infeasible, or it is unbounded, or it has an optimal solution.

Property 2:
In a linear programming problem each local optimal solution is also a global optimal solution.

Property 3:
If a linear programming problem has an optimal solution and has at least one extreme point, then there exists an optimal solution that is an extreme point.

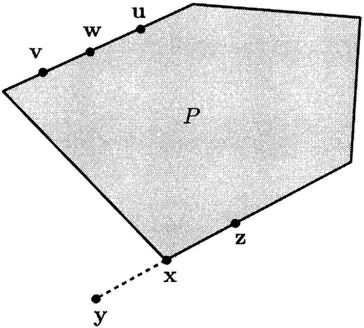
## 2.2 Extreme points, vertices, and basic feasible solutions

# Extreme points, vertices, and basic feasible solutions

- We observed in Section 1.4 that an optimal solution to a linear programming problem tends to occur at a "corner" of the polyhedron over which we are optimizing.
- We now introduce three different ways of defining the concept of a "corner".
- We then show that all three definitions are equivalent.

# Extreme points

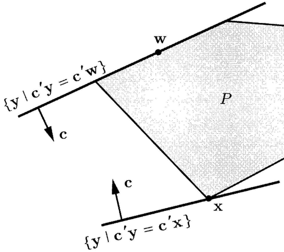
**Definition 2.6**
Let $P$ be a polyhedron. A vector $x \in P$ is an extreme point of $P$ if we cannot find two vectors $y, z \in P$, both different from $x$, and a scalar $\lambda \in [0, 1]$, such that $x = \lambda y + (1 - \lambda)z$.




- This definition is entirely geometric: It depends on $P$, but it does not depend on the (algebraic) representation of $P$.

# Vertices

**Definition 2.7**
Let $P$ be a polyhedron. A vector $x \in P$ is a vertex of $P$ if there exists some $c$ such that $c'x < c'y$ for all $y$ satisfying $y \in P$ and $y \neq x$.




- A vertex of a polyhedron $P$ is the unique optimal solution to some linear programming problem with feasible set $P$.

- Also this definition is entirely geometric.
- We would like to have a definition which reduces to an algebraic test.

# Active constraints in $x^* \in \mathbb{R}^n$

Consider a polyhedron $P \subset \mathbb{R}^n$ defined in terms of the linear equality and inequality constraints

$$
\begin{aligned}
a_i' x &\geq b_i, \quad i \in M_1, \\
a_i' x &\leq b_i, \quad i \in M_2, \\
a_i' x &= b_i, \quad i \in M_3,
\end{aligned}
$$

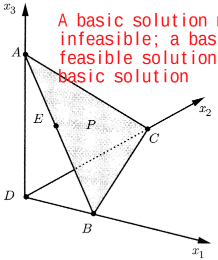
where $M_1$, $M_2$, and $M_3$ are finite index sets, each $a_i$ is a vector in $\mathbb{R}^n$, and each $b_i$ is a scalar.

A,B,C are basic feasible solutions

A basic solution might be infeasible; a basic feasible solution is also a basic solution

**Definition 2.8**

If a vector $x^*$ satisfies $a_i' x^* = b_i$ for $i$ in $M_1$, $M_2$, or $M_3$, we say that constraint $i$ is active at $x^*$.




Example: $P = \{(x_1, x_2, x_3 \mid x_1 + x_2 + x_3 = 1,\ x_1, x_2, x_3 \geq 0)\}$.

# Basic (feasible) solutions

**Definition 2.9**

Consider a polyhedron $P$ defined by linear equality and inequality constraints, and let $x^*$ be an element of $\mathbb{R}^n$.

(a) The vector $x^*$ is a basic solution if:
   (i) All equality constraints are active;
   (ii) Out of the constraints that are active at $x^*$, there are $n$ of them that are linearly independent.

(b) If $x^*$ is a basic solution that satisfies all of the constraints, we say that it is a basic feasible solution.

- We say that certain constraints $a_i x \sim b_i$ are linearly independent, meaning that the corresponding vectors $a_i$ are linearly independent.

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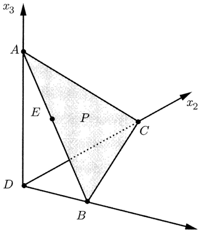


Example:

$$P = \{(x_1, x_2, x_3) : x_1 + x_2 + x_3 = 1,\ x_1, x_2, x_3 \geq 0\}.$$

# Basic (feasible) solutions

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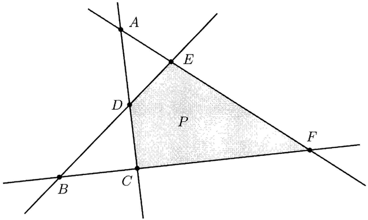



F point in on two linearly independent constraints that are also active;
B is a basic solution, but is not feasible

# Basic (feasible) solutions

- If the number $m$ of constraints used to define a polyhedron $P \subset \mathbb{R}^n$ is less than $n$, the number of active constraints at any given point must also be less than $n$, and there are no basic or basic feasible solutions.

# Equivalence of definitions

- We have given three different definitions that are meant to capture the same concept.
- Two of them are geometric (extreme point, vertex) and the third is algebraic (basic feasible solution).
- All three definitions are equivalent and, for this reason, the three terms can be used interchangeably.

**Theorem 2.3**
Let $P$ be a nonempty polyhedron and let $x^* \in P$. Then, the following are equivalent:

(a) $x^*$ is a vertex;

(b) $x^*$ is an extreme point;

(c) $x^*$ is a basic feasible solution.

# Equivalence of definitions

- To prove Theorem 2.3 we need the following result.

**Theorem 2.2 (edited)**
Let $a_i'x = b_i$, $i \in I$, be constraints in $\mathbb{R}^n$. The following are equivalent:

(a) There exist $n$ constraints among $a_i'x = b_i$, $i \in I$, which are linearly independent.

(b) The span of the vectors $a_i, i \in I$, is all of $\mathbb{R}^n$, that is, every element of $\mathbb{R}^n$ can be expressed as a linear combination of the vectors $a_i, i \in I$.

(c) The system of equations $a_i'x = b_i$, $i \in I$, has a unique solution.

Proof: Exercise. You can prove Theorem 2.2 using basic facts from linear algebra (Theorem 1.3(a) in Section 1.5).

Now let's prove Theorem 2.3!

# Equivalence of definitions

- Theorem 2.3 states that a vector is a basic feasible solution if and only if it is an extreme point.
- We have seen that the definition of an extreme point does not depend on the representation of a polyhedron.
- We conclude that the property of being a basic feasible solution is also independent of the representation of the polyhedron.

# Equivalence of definitions

- Theorem 2.3 states that a vector is a basic feasible solution if and only if it is an extreme point.
- We have seen that the definition of an extreme point does not depend on the representation of a polyhedron.
- We conclude that the property of being a basic feasible solution is also independent of the representation of the polyhedron.

- Question: Does the property of being a basic solution depend on the representayion of the polyhedron?

# Equivalence of definitions

**Corollary 2.1**
Given a finite number of linear inequality constraints, there can only be a finite number of basic or basic feasible solutions.

**Proof.**

- Consider a system of $m$ linear inequality constraints in $\mathbb{R}^n$.
- At any basic solution, there are $n$ linearly independent active constraints that define it.
- Therefore, the number of basic solutions is bounded above by the number of ways that we can choose $n$ constraints out of $m$.
- This number is finite. □

# Equivalence of definitions

> **Corollary 2.1**
> Given a finite number of linear inequality constraints, there can only be a finite number of basic or basic feasible solutions.

- Although the number of basic feasible solutions is guaranteed to be finite, it can be very large.
- For example, the unit cube

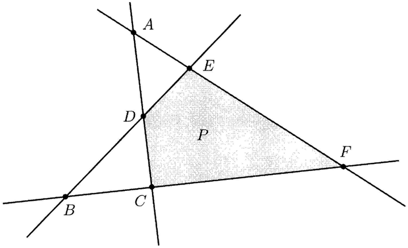
$$\{x \in \mathbb{R}^n \mid 0 \leq x_i \leq 1, \ i = 1, \ldots, n\}$$

is defined by $2n$ constraints, but has $2^n$ basic feasible solutions.

# Adjacent basic solutions

# Adjacent basic solutions

- Two distinct basic solutions to a set of linear constraints in $\mathbb{R}^n$ are said to be adjacent if we can find $n-1$ linearly independent constraints that are active at both of them.
- If two adjacent basic solutions are also feasible, then the line segment that joins them is called an edge of the feasible set.



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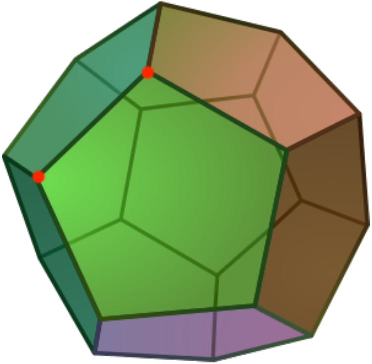

# 2.3 Polyhedra in standard form

# Polyhedra in standard form

- The definition of a basic solution refers to general polyhedra.
- We will now specialize to polyhedra in standard form.

- Let
$$P = \{x \in \mathbb{R}^n \mid Ax = b,\ x \geq 0\}$$
be a polyhedron in standard form.
- Let the dimensions of $A$ be $m \times n$.

# The full row rank assumption on $A$

- In most of our discussion of standard form problems, we will make the assumption that the $m$ rows of the matrix $A$ are linearly independent.
- We now see that when $P$ is nonempty, linearly dependent rows of $A$ correspond to redundant constraints that can be discarded.
- Therefore, our linear independence assumption can be made without loss of generality.

# The full row rank assumption on $A$

**Theorem 2.5**
Let $P = \{x \mid Ax = b,\ x \geq 0\}$ be a nonempty polyhedron, where $A$ is a matrix of dimensions $m \times n$, with rows $a_1', \ldots, a_m'$. Suppose that $\text{rank}(A) = k < m$ and that the rows $a_{i_1}', \ldots, a_{i_k}'$ are linearly independent. Consider the polyhedron

$$Q = \{x \mid a_{i_1}'x = b_{i_1}, \ldots, a_{i_k}'x = b_{i_k},\ x \geq 0\}.$$

Then $Q = P$.

Proof: Easy exercise.

# Example 2.3

- Consider the nonempty polyhedron defined by

$$\begin{aligned} 2x_1 + x_2 + x_3 &= 2 \\ x_1 + x_2 &= 1 \\ x_1 + x_3 &= 1 \\ x_1, x_2, x_3 &\geq 0. \end{aligned}$$

# Example 2.3

- Consider the nonempty polyhedron defined by

$$\begin{aligned} 2x_1 + x_2 + x_3 &= 2 \\ x_1 + x_2 &= 1 \\ x_1 + x_3 &= 1 \\ x_1, x_2, x_3 &\geq 0. \end{aligned}$$

- The corresponding matrix $A$ has rank two.
- This is because the last two rows $(1, 1, 0)$ and $(1, 0, 1)$ are linearly independent, but the first row $(2, 1, 1)$ is equal to the sum of the other two.

# Example 2.3

- Consider the nonempty polyhedron defined by

$$\begin{aligned} \sout{2x_1 + x_2 + x_3 = 2} \\ x_1 + x_2 = 1 \\ x_1 + x_3 = 1 \\ x_1, x_2, x_3 \geq 0. \end{aligned}$$

- The corresponding matrix $A$ has rank two.
- This is because the last two rows $(1, 1, 0)$ and $(1, 0, 1)$ are linearly independent, but the first row $(2, 1, 1)$ is equal to the sum of the other two.
- Thus, the first constraint is redundant and after it is eliminated, we still have the same polyhedron.

# The full row rank assumption on $A$

- Notice that the polyhedron $Q$ in Theorem 2.5 is in standard form like $P$:

$$Q = \{x \mid Dx = f,\ x \geq 0\}.$$

- Moreover, $D$ is a $k \times n$ submatrix of $A$, with rank equal to $k$.
- We conclude that as long as the feasible set is nonempty, a linear programming problem in standard form can be reduced to an equivalent standard form problem (with the same feasible set) in which the equality constraints are linearly independent.

# Polyhedra in standard form: basic solutions

- Let's get back to our task of specializing the definition of a basic solution to polyhedra in standard form.
- Let
$$P = \{x \in \mathbb{R}^n \mid Ax = b,\ x \geq 0\}$$
be a polyhedron in standard form.
- Let the dimensions of $A$ be $m \times n$.
- We can now assume, without loss of generality, that the $m$ rows of the matrix $A$ are linearly independent.
- Since the rows are $n$-dimensional, this requires that $m \leq n$.

# Polyhedra in standard form: basic solutions

- Recall that at any basic solution, there must be $n$ linearly independent constraints that are active.
- Furthermore, every basic solution must satisfy the equality constraints $Ax = b$, which provides us with $m$ active constraints; these are linearly independent because of our assumption on the rows of $A$.
- In order to obtain a total of $n$ active constraints, we need to choose $n - m$ of the variables $x_i$ and set them to zero, which makes the corresponding nonnegativity constraints $x_i \geq 0$ active.
- However, for the resulting set of $n$ active constraints to be linearly independent, the choice of these $n - m$ variables is not entirely arbitrary.

# Polyhedra in standard form: basic solutions

**Theorem 2.4**

Consider the constraints $Ax = b$ and $x \geq 0$ and assume that the $m \times n$ matrix $A$ has linearly independent rows. A vector $x \in \mathbb{R}^n$ is a basic solution if and only if we have $Ax = b$, and there exist distinct indices $B(1), \ldots, B(m) \in \{1, \ldots, n\}$ such that:

(a) The columns $A_{B(1)}, \ldots, A_{B(m)}$ of $A$ are linearly independent;

(b) If $i \neq B(1), \ldots, B(m)$, then $x_i = 0$.

Proof idea:

Active constraints:
$$\begin{matrix} m \\ n-m \end{matrix} \overset{n}{\begin{pmatrix} A \\ 0 \mid I \end{pmatrix}} = \begin{matrix} m \\ n-m \end{matrix} \begin{pmatrix} \overset{m}{B} & \overset{n-m}{C} \\ 0 & I \end{pmatrix}$$

# Polyhedra in standard form: basic solutions

In view of Theorem 2.4, all basic solutions to a standard form polyhedron can be constructed according to the following procedure.

**Procedure for constructing basic solutions**

1. Choose $m$ linearly independent columns $A_{B(1)}, \dots, A_{B(m)}$.
2. Let $x_i = 0$ for all $i \neq B(1), \dots, B(m)$.
3. Solve the system of $m$ equations $Ax = b$ for the unknowns $x_{B(1)}, \dots, x_{B(m)}$.

# Polyhedra in standard form: basic solutions

We can use a similar procedure to construct all basic feasible solutions to a standard form polyhedron.

- If a basic solution constructed according to this procedure is nonnegative, then it is feasible, and it is a basic feasible solution.
- Conversely, since every basic feasible solution is a basic solution, it can be obtained from this procedure.

# Example 2.1

Let the constraint $Ax = b$ be of the form

$$\begin{bmatrix} 1 & 1 & 2 & 1 & 0 & 0 & 0 \\ 0 & 1 & 6 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} x = \begin{bmatrix} 8 \\ 12 \\ 4 \\ 6 \end{bmatrix}.$$

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- The columns $A_4, A_5, A_6, A_7$ are linearly independent.
- The corresponding basic solution is

$$x = (0, 0, 0, 8, 12, 4, 6).$$

- $x \geq 0 \quad \Rightarrow \quad x$ is a basic feasible solution.

# Example 2.1

Let the constraint $Ax = b$ be of the form

$$\begin{bmatrix} 1 & 1 & 2 & 1 & 0 & 0 & 0 \\ 0 & 1 & 6 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} x = \begin{bmatrix} 8 \\ 12 \\ 4 \\ 6 \end{bmatrix}.$$

- The columns $A_3, A_5, A_6, A_7$ are linearly independent.
- The corresponding basic solution is

$$x = (0, 0, 4, 0, -12, 4, 6).$$

- $x_5 = -12 < 0 \quad \Rightarrow \quad x$ is not feasible.

# Polyhedra in standard form: basic solutions

- If $x$ is a basic solution, the variables

$$x_{B(1)}, \dots, x_{B(m)}$$

are called basic variables; the remaining variables are called nonbasic.

- Similarly, $B(1), \dots, B(m)$ are called basic indices, and $\{1, \dots, n\} \setminus \{B(1), \dots, B(m)\}$ are called nonbasic indices.
- The columns

$$A_{B(1)}, \dots, A_{B(m)}$$

are called the basic columns and, since they are linearly independent, they form a basis of $\mathbb{R}^m$.

# Polyhedra in standard form: basic solutions

- By arranging the $m$ basic columns next to each other, we obtain an $m \times m$ matrix $B$, called a basis matrix:

$$B = \begin{bmatrix} A_{B(1)} & A_{B(2)} & \cdots & A_{B(m)} \end{bmatrix}.$$

- Note that a basis matrix $B$ is invertible because the basic columns are linearly independent.
- We can similarly define a vector $x_B$ with the values of the basic variables:

$$x_B = \begin{bmatrix} x_{B(1)} \\ \vdots \\ x_{B(m)} \end{bmatrix}.$$

- The basic variables are determined by solving the equation $Bx_B = b$ whose unique solution is given by

$$x_B = B^{-1}b.$$

# Example 2.1

$$\begin{bmatrix} 1 & 1 & 2 & 1 & 0 & 0 & 0 \\ 0 & 1 & 6 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} x = \begin{bmatrix} 8 \\ 12 \\ 4 \\ 6 \end{bmatrix}.$$

- We chose columns $A_4, A_5, A_6, A_7$ and obtained the basic feasible solution $x = (0, 0, 0, 8, 12, 4, 6)$.
- Basic indices: $B(1) = 4$, $B(2) = 5$, $B(3) = 6$, $B(4) = 7$.
- Basic variables: $x_{B(1)} = x_4$, $x_{B(2)} = x_5$, $x_{B(3)} = x_6$, $x_{B(4)} = x_7$.
- Basic columns: $A_{B(1)} = A_4$, $A_{B(2)} = A_5$, $x_{B(3)} = A_6$, $x_{B(4)} = A_7$.

# Example 2.1

$$\begin{bmatrix} 1 & 1 & 2 & 1 & 0 & 0 & 0 \\ 0 & 1 & 6 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} x = \begin{bmatrix} 8 \\ 12 \\ 4 \\ 6 \end{bmatrix}.$$

- Basis matrix $B$ and vector $x_B$:

$$B = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad x_B = \begin{bmatrix} x_4 \\ x_5 \\ x_6 \\ x_7 \end{bmatrix}.$$

- Note that $B_i = A_{B(i)}$ for every $i = 1, \ldots, m$.

# Different bases

- We say that two bases are distinct or different if they involve different sets $\{B(1), \ldots, B(m)\}$ of basic indices.
- If two bases involve the same set of indices in a different order, they will be viewed as one and the same basis.

# Example 2.1

Suppose there is an eighth column $A_8 = A_7$.

$$\begin{bmatrix} 1 & 1 & 2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 6 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix} x = \begin{bmatrix} 8 \\ 12 \\ 4 \\ 6 \end{bmatrix}.$$

- The two sets of columns $\{A_3, A_5, A_6, A_7\}$ and $\{A_3, A_5, A_6, A_8\}$ coincide.
- The corresponding sets of basic indices $\{3, 5, 6, 7\}$ and $\{3, 5, 6, 8\}$ are different.
- We have two different bases.

# Example 2.1

Suppose there is an eighth column $A_8 = A_7$.

$$\begin{bmatrix} 1 & 1 & 2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 6 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix} x = \begin{bmatrix} 8 \\ 12 \\ 4 \\ 6 \end{bmatrix}.$$

- The two sets of columns $\{A_3, A_5, A_6, A_7\}$ and $\{A_3, A_5, A_6, A_8\}$ coincide.
- The corresponding sets of basic indices $\{3, 5, 6, 7\}$ and $\{3, 5, 6, 8\}$ are different.
- We have two different bases.

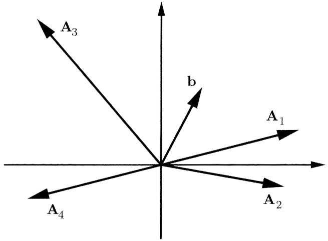
# Polyhedra in standard form: basic solutions

Intuitive view of basic solutions.

- Recall our interpretation of the constraint

$$Ax = b \quad \Leftrightarrow \quad \sum_{i=1}^{n} A_i x_i = b$$

  as a requirement to synthesize the vector $b \in \mathbb{R}^n$ using the resource vectors $A_i$ (Section 1.1).
- In a basic solution, we use only $m$ of the resource vectors, those associated with the basic variables.
- In a basic feasible solution, this is accomplished using a nonnegative amount of each basic vector.




Question: In this example $m = 2$. Which are the bases that yield basic feasible solutions?

# Correspondence of bases and basic solutions

# Correspondence of bases and basic solutions

- A basis uniquely determines a basic solution, thus different basic solutions must correspond to different bases.
- However, two different bases may lead to the same basic solution.
- Example: If we have $b = 0$, then every basis matrix leads to the same basic solution, namely, the zero vector.
- This phenomenon will have some important algorithmic implications.

# Adjacent basic solutions and adjacent bases

# Adjacent basic solutions and adjacent bases

- Recall that two distinct basic solutions are said to be adjacent if there are $n-1$ linearly independent constraints that are active at both of them.

- For standard form problems, we define two bases to be adjacent if they share all but one basic columns.

Exercise: Show that:

1. Adjacent basic solutions can always be obtained from two adjacent bases.
2. Conversely, if two adjacent bases lead to distinct basic solutions, then the latter are adjacent.

(Hint: Use proof of Theorem 2.4.)

# Example 2.2

In Example 2.1 we considered constraint $Ax = b$ of the form

$$\begin{bmatrix} 1 & 1 & 2 & 1 & 0 & 0 & 0 \\ 0 & 1 & 6 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} x = \begin{bmatrix} 8 \\ 12 \\ 4 \\ 6 \end{bmatrix}.$$

# Example 2.2

In Example 2.1 we considered constraint $Ax = b$ of the form

$$\begin{bmatrix} 1 & 1 & 2 & 1 & 0 & 0 & 0 \\ 0 & 1 & 6 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} x = \begin{bmatrix} 8 \\ 12 \\ 4 \\ 6 \end{bmatrix}.$$

- The bases $\{A_4, A_5, A_6, A_7\}$ and $\{A_3, A_5, A_6, A_7\}$ are adjacent because all but one columns are the same.
- The corresponding basic solutions $x = (0, 0, 0, 8, 12, 4, 6)$ and $x = (0, 0, 4, 0, -12, 4, 6)$ are adjacent: we have $n = 7$ and a total of six common linearly independent active constraints: the four equality constraints, $x_1 \geq 0$, and $x_2 \geq 0$.

# Example 2.2

In Example 2.1 we considered constraint $Ax = b$ of the form

$$\begin{bmatrix} 1 & 1 & 2 & 1 & 0 & 0 & 0 \\ 0 & 1 & 6 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} x = \begin{bmatrix} 8 \\ 12 \\ 4 \\ 6 \end{bmatrix}.$$

- The bases $\{A_4, A_5, A_6, A_7\}$ and $\{A_3, A_5, A_6, A_7\}$ are adjacent because all but one columns are the same.
- The corresponding basic solutions $x = (0, 0, 0, 8, 12, 4, 6)$ and $x = (0, 0, 4, 0, -12, 4, 6)$ are adjacent: we have $n = 7$ and a total of six common linearly independent active constraints: the four equality constraints, $x_1 \geq 0$, and $x_2 \geq 0$.

## 2.4 Degeneracy

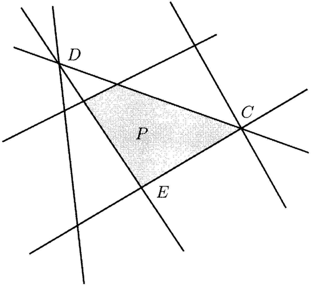
# Degeneracy

- We now consider again general polyhedra.
- According to our definition, at a basic solution, we must have $n$ linearly independent active constraints.
- This allows for the possibility that the number of active constraints is greater than $n$.
- In this case, we say that we have a degenerate basic solution.

**Definition 2.10**
A basic solution $x \in \mathbb{R}^n$ is said to be degenerate if more than $n$ of the constraints are active at $x$.

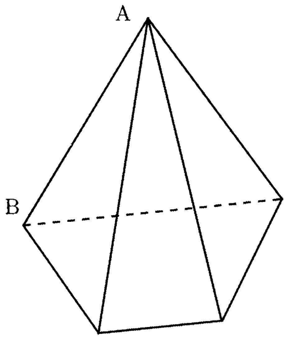
# Degeneracy

- If $n = 2$, a degenerate basic solution is at the intersection of three or more lines.



# Degeneracy

- If $n = 3$, a degenerate basic solution is at the intersection of four or more planes.




- The presence of degeneracy can strongly affect the behavior of linear programming algorithms.

# Example 2.4

Consider the polyhedron $P$ defined by the constraints

$$
\begin{aligned}
x_1 + x_2 + 2x_3 &\leq 8 && (1)\\
x_2 + 6x_3 &\leq 12 && (2)\\
x_1 &\leq 4 && (3)\\
x_2 &\leq 6 && (4)\\
x_1, x_2, x_3 &\geq 0.
\end{aligned}
$$

# Example 2.4

Consider the polyhedron $P$ defined by the constraints

$$
\begin{aligned}
x_1 + x_2 + 2x_3 &\leq 8 \quad (1)\\
x_2 + 6x_3 &\leq 12 \quad (2)\\
x_1 &\leq 4 \quad (3)\\
x_2 &\leq 6 \quad (4)\\
x_1, x_2, x_3 &\geq 0.
\end{aligned}
$$

- Consider $x = (2, 6, 0)$.

# Example 2.4

Consider the polyhedron $P$ defined by the constraints

$$\begin{aligned} x_1 + x_2 + 2x_3 &\leq 8 && (1) \\ x_2 + 6x_3 &\leq 12 && (2) \\ x_1 &\leq 4 && (3) \\ x_2 &\leq 6 && (4) \\ x_1, x_2, x_3 &\geq 0. \end{aligned}$$

- Consider $x = (2, 6, 0)$.
  There are exactly three active and linearly independent constraints: (**??**), (**??**), and $x_3 \geq 0$.

# Example 2.4

Consider the polyhedron $P$ defined by the constraints

$$\begin{aligned} x_1 + x_2 + 2x_3 &\leq 8 && (1) \\ x_2 + 6x_3 &\leq 12 && (2) \\ x_1 &\leq 4 && (3) \\ x_2 &\leq 6 && (4) \\ x_1, x_2, x_3 &\geq 0. \end{aligned}$$

- Consider $x = (2, 6, 0)$.
  There are exactly three active and linearly independent constraints: (**??**), (**??**), and $x_3 \geq 0$.
  Thus it is a nondegenerate basic feasible solution.

# Example 2.4

Consider the polyhedron $P$ defined by the constraints

$$x_1 + x_2 + 2x_3 \leq 8 \quad (1)$$
$$x_2 + 6x_3 \leq 12 \quad (2)$$
$$x_1 \leq 4 \quad (3)$$
$$x_2 \leq 6 \quad (4)$$
$$x_1, x_2, x_3 \geq 0.$$

- Consider $x = (2, 6, 0)$.
  There are exactly three active and linearly independent constraints: (**??**), (**??**), and $x_3 \geq 0$.
  Thus it is a nondegenerate basic feasible solution.
- Consider $x = (4, 0, 2)$.

# Example 2.4

Consider the polyhedron $P$ defined by the constraints

$$
\begin{aligned}
x_1 + x_2 + 2x_3 &\leq 8 && (1)\\
x_2 + 6x_3 &\leq 12 && (2)\\
x_1 &\leq 4 && (3)\\
x_2 &\leq 6 && (4)\\
x_1, x_2, x_3 &\geq 0.
\end{aligned}
$$

- Consider $x = (2, 6, 0)$.
  There are exactly three active and linearly independent constraints: (**??**), (**??**), and $x_3 \geq 0$.
  Thus it is a nondegenerate basic feasible solution.
- Consider $x = (4, 0, 2)$.
  There are four active constraints: (**??**), (**??**), (**??**), and $x_2 \geq 0$, and three of them are linearly independent.

# Example 2.4

Consider the polyhedron $P$ defined by the constraints

$$x_1 + x_2 + 2x_3 \leq 8 \quad (1)$$
$$x_2 + 6x_3 \leq 12 \quad (2)$$
$$x_1 \leq 4 \quad (3)$$
$$x_2 \leq 6 \quad (4)$$
$$x_1, x_2, x_3 \geq 0.$$

- Consider $x = (2, 6, 0)$.
  There are exactly three active and linearly independent constraints: (**??**), (**??**), and $x_3 \geq 0$.
  Thus it is a nondegenerate basic feasible solution.
- Consider $x = (4, 0, 2)$.
  There are four active constraints: (**??**), (**??**), (**??**), and $x_2 \geq 0$, and three of them are linearly independent.
  Thus it is a degenerate basic feasible solution.

# Degeneracy in standard form polyhedra

# Degeneracy in standard form polyhedra

- At a basic solution of a polyhedron in standard form, the $m$ equality constraints are always active.
- Therefore, having more than $n$ active constraints is the same as having more than $n - m$ variables at zero level.
- This leads us to the next definition which is a special case of Definition 2.10.

**Definition 2.11**
Consider the standard form polyhedron

$$P = \{x \in \mathbb{R}^n \mid Ax = b, \ x \geq 0\}$$

and let $x$ be a basic solution. Let $m$ be the number of rows of $A$. The vector $x$ is a degenerate basic solution if more than $n - m$ of the components of $x$ are zero.

# Example 2.5

Consider the polyhedron $P = \{x \mid Ax = b,\ x \geq 0\}$, where

$$A = \begin{bmatrix} 1 & 1 & 2 & 1 & 0 & 0 & 0 \\ 0 & 1 & 6 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad b = \begin{bmatrix} 8 \\ 12 \\ 4 \\ 6 \end{bmatrix}.$$

# Example 2.5

Consider the polyhedron $P = \{x \mid Ax = b,\ x \geq 0\}$, where

$$A = \begin{bmatrix} 1 & 1 & 2 & 1 & 0 & 0 & 0 \\ 0 & 1 & 6 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad b = \begin{bmatrix} 8 \\ 12 \\ 4 \\ 6 \end{bmatrix}.$$

- Consider the basis consisting of the linearly independent columns $A_1, A_2, A_3, A_7$.
- The corresponding basic (feasible) solution is $x = (4, 0, 2, 0, 0, 0, 6)$.
- It is degenerate, because we have a total of four variables that are zero, whereas $n - m = 7 - 4 = 3$.

# Example 2.5

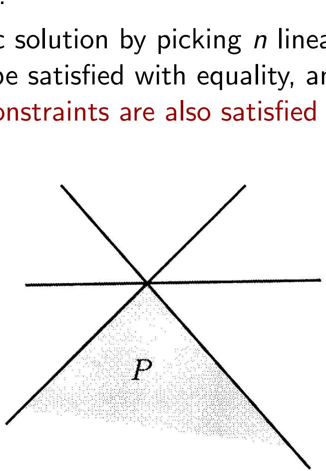
Consider the polyhedron $P = \{x \mid Ax = b,\ x \geq 0\}$, where

$$A = \begin{bmatrix} 1 & 1 & 2 & 1 & 0 & 0 & 0 \\ 0 & 1 & 6 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad b = \begin{bmatrix} 8 \\ 12 \\ 4 \\ 6 \end{bmatrix}.$$

- Consider now the adjacent basis consisting of the linearly independent columns $A_1, A_3, A_4, A_7$.
- The corresponding basic (feasible) solution is again $x = (4, 0, 2, 0, 0, 0, 6)$.

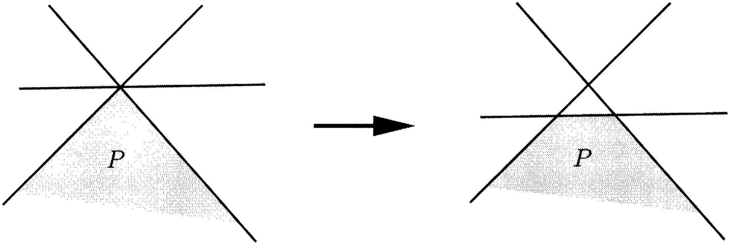
# Degeneracy in standard form polyhedra

- This example suggests that we can think of degeneracy in the following terms.
- We pick a basic solution by picking $n$ linearly independent constraints to be satisfied with equality, and we realize that certain other constraints are also satisfied with equality.



# Degeneracy in standard form polyhedra

- If the entries of $A$ or $b$ were chosen at random, this would almost never happen.
- Also, if the coefficients of the active constraints are slightly perturbed, degeneracy can disappear.




- In practical problems, however, the entries of $A$ and $b$ often have a special (nonrandom) structure, and degeneracy is common.

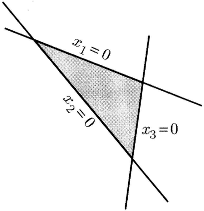
# Degeneracy in standard form polyhedra

To visualize degeneracy in standard form polyhedra, we assume that $n - m = 2$.

Example:

$n = 3$

$m = 1$




- At a nondegenerate basic solution, exactly $n - m$ of the constraints $x_i \geq 0$ are active and the corresponding variables are nonbasic.

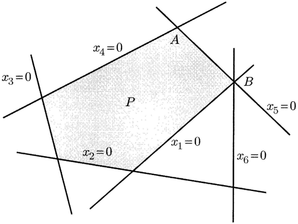
# Degeneracy in standard form polyhedra

To visualize degeneracy in standard form polyhedra, we assume that $n - m = 2$.

Example:

$n = 6$

$m = 4$




- At a nondegenerate basic solution, exactly $n - m$ of the constraints $x_i \geq 0$ are active and the corresponding variables are nonbasic.

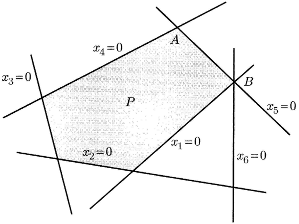
# Degeneracy in standard form polyhedra

To visualize degeneracy in standard form polyhedra, we assume that $n - m = 2$.

Example:

$n = 6$

$m = 4$



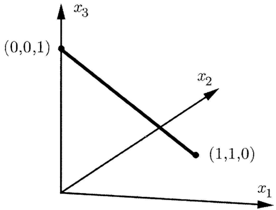

- At a degenerate basic solution, more than $n - m$ of the constraints $x_i \geq 0$ are active, and there are several ways of choosing which $n - m$ variables to call nonbasic; in that case, there are several bases corresponding to that same basic solution.

Degeneracy is not a purely geometric property

# Degeneracy is not a purely geometric property

- Degeneracy of basic feasible solutions may depend on the particular representation of a polyhedron.
- Let's see an example.

# Degeneracy is not a purely geometric property

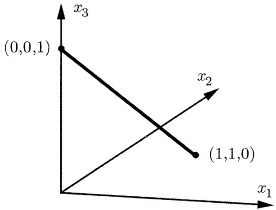



Example: Consider the standard form polyhedron

$$P = \{(x_1, x_2, x_3) \mid x_1 - x_2 = 0,\ x_1 + x_2 + 2x_3 = 2,\ x_1, x_2, x_3 \geq 0\}.$$

- $n = 3,\ m = 2,\ n - m = 1.$
- $(1, 1, 0)$ is a nondegenerate basic feasible solution because only one variable is zero.
- $(0, 0, 1)$ is a degenerate basic feasible solution because two variables are zero.

# Degeneracy is not a purely geometric property

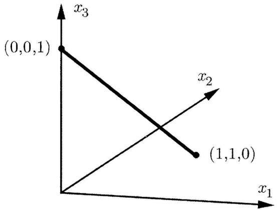



The same polyhedron can also be described in the form

$$P = \{(x_1, x_2, x_3) \mid x_1 - x_2 = 0,\ x_1 + x_2 + 2x_3 = 2,\ x_1, x_3 \geq 0\}.$$

- Note that this polyhedron is not in standard form.
- The vector $(0, 0, 1)$ is now a nondegenerate basic feasible solution, because there are only three active constraints.

# Degeneracy is not a purely geometric property




The same polyhedron can also be described in the form

$$P = \{(x_1, x_2, x_3) \mid x_1 - x_2 = 0,\ x_1 + x_2 + 2x_3 = 2,\ x_1, x_3 \geq 0\}.$$

- Note that this polyhedron is not in standard form.
- The vector $(0, 0, 1)$ is now a nondegenerate basic feasible solution, because there are only three active constraints.

Question: Can you construct a polytope with a basic feasible solution that is degenerate in every possible representation?

# Degeneracy is not a purely geometric property

- Example: consider a standard form polyhedron

$$P = \{x \mid Ax = b,\ x \geq 0\}.$$

- Let $x^*$ be a nondegenerate basic feasible solution.
- Exactly $n - m$ of the variables $x_i^*$ are equal to zero.

# Degeneracy is not a purely geometric property

- Example: consider a standard form polyhedron

$$P = \{x \mid Ax = b,\ x \geq 0\}.$$

- Let $x^*$ be a nondegenerate basic feasible solution.
- Exactly $n - m$ of the variables $x_i^*$ are equal to zero.

- We now represent $P$ in the form

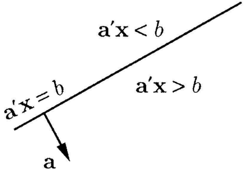
$$P = \{x \mid Ax \geq b,\ -Ax \geq -b,\ x \geq 0\}.$$

- Active constraints at $x^*$:
  - $n - m$ variables set to zero,
  - all $m$ inequality constraints from $Ax \geq b$,
  - all $m$ inequality constraints from $-Ax \geq -b$.
- Total active constraints at $x^*$: $n - m + 2m = n + m$.
  $\Rightarrow x^*$ is degenerate.
- Under the second representation every basic feasible solution is degenerate.

# 2.5 Existence of extreme points

# Existence of extreme points

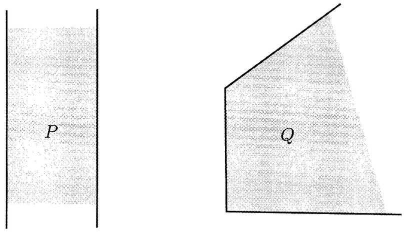
- We will see necessary and sufficient conditions for a polyhedron to have at least one extreme point.
- Observe that not every polyhedron has this property.
- Example: If $n \geq 2$, a halfspace in $\mathbb{R}^n$ is a polyhedron without extreme points.




- More generally, as argued in Section 2.2, if the matrix $A$ has fewer than $n$ rows, then the polyhedron $\{x \in \mathbb{R}^n \mid Ax \geq b\}$ does not have a basic feasible solution.

# Existence of extreme points

- The existence of an extreme point depends on whether a polyhedron contains an infinite line or not.




> **Definition 2.12**
> A polyhedron $P \subset \mathbb{R}^n$ contains a line if there exists a vector $x \in P$ and a nonzero vector $d \in \mathbb{R}^n$ such that $x + \lambda d \in P$ for all $\lambda \in \mathbb{R}$.

Exercise: Show that if $P$ contains a line $\{\bar{x} + \lambda d \mid \lambda \in \mathbb{R}\}$, then $P$ contains all lines $\{x + \lambda d \mid \lambda \in \mathbb{R}\}$ for every $x \in P$.

# Existence of extreme points

**Theorem 2.6**

Suppose that the polyhedron

$$P = \{x \in \mathbb{R}^n \mid a_i' x \geq b_i,\ i = 1, \ldots, m\}$$

is nonempty. Then, the following are equivalent:

(a) The polyhedron $P$ has at least one extreme point.

(b) The polyhedron $P$ does not contain a line.

(c) There exist $n$ vectors out of the family $a_1, \ldots, a_m$, which are linearly independent.

Proof idea:

- $(a) \Rightarrow (c)$: Immediate, from definition of basic solution.
- $(c) \Rightarrow (b)$: By contradiction. Assume $P$ contains a line $\{x + \lambda d \mid \lambda \in \mathbb{R}\}$. Then $d$ must be orthogonal to each vector $a_i$.

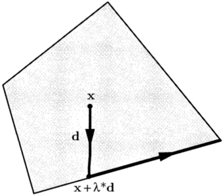
# Existence of extreme points

(a) The polyhedron $P$ has at least one extreme point.
(b) The polyhedron $P$ does not contain a line.
(c) There exist $n$ vectors out of the family $a_1, \dots, a_m$, which are linearly independent.

Proof idea:

- $(a) \Rightarrow (c)$: Immediate, from definition of basic solution.
- $(c) \Rightarrow (b)$: By contradiction. Assume $P$ contains a line $\{x + \lambda d \mid \lambda \in \mathbb{R}\}$. Then $d$ must be orthogonal to each vector $a_i$.
- $(b) \Rightarrow (a)$: Constructive:
  1. Start from $x \in P$ and move along a direction $d$ where all active constraints remain active.
  2. Stop when a new constraint is about to be violated.
  3. Now there is one more linearly independent active constraint. Repeat.

  Exercise: Write a complete proof.



# Existence of extreme points

Examples of polyhedra that do not contain a line:

- Polytopes. Why?
- The positive orthant $\{x \mid x \geq 0\}$. Why?
- Polyhedra in standard form. Why?

Theorem 2.6 then implies:

**Corollary 2.2**
Every nonempty bounded polyhedron and every nonempty polyhedron in standard form has at least one basic feasible solution.

# 2.6 Optimality of extreme points

# Optimality of extreme points

- We have established the conditions for the existence of extreme points.
- We now confirm the intuition developed in Chapter 1:

**Theorem 2.7**
Consider the linear programming problem of minimizing $c'x$ over a polyhedron $P$. Suppose that $P$ has at least one extreme point and that there exists an optimal solution. Then, there exists an optimal solution which is an extreme point of $P$.

Let's prove it!

- We have seen that polyhedra in standard form, as well as bounded polyhedra, have extreme points.
- In these cases, Theorem 2.7 implies that if there exists an optimal solution, then there exists an extreme point that is optimal.

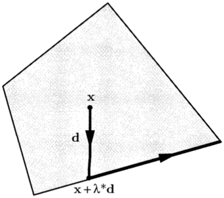
# Optimality of extreme points

- The next result is stronger than Theorem 2.7.
- It shows that the existence of an optimal solution can be taken for granted, as long as the optimal cost is not $-\infty$.

**Theorem 2.8**
Consider the linear programming problem of minimizing $c'x$ over a polyhedron $P$. Suppose that $P$ has at least one extreme point. Then, either the optimal cost is equal to $-\infty$, or there exists an extreme point which is optimal.

Proof idea:

- Similar to the proof of Theorem 2.6 $(b) \Rightarrow (a)$. The difference is that we also make sure that the cost does not increase.
- This shows that $\forall x \in P$, either the optimal cost is $-\infty$, or $\exists$ extreme point with cost $\leq$.
- Exercise: Write a complete proof.



# Optimality of extreme points

- The next result is stronger than Theorem 2.7.
- It shows that the existence of an optimal solution can be taken for granted, as long as the optimal cost is not $-\infty$.

**Theorem 2.8**
Consider the linear programming problem of minimizing $c'x$ over a polyhedron $P$. Suppose that $P$ has at least one extreme point. Then, either the optimal cost is equal to $-\infty$, or there exists an extreme point which is optimal.

- For a general linear programming problem, if the feasible set has no extreme points, then Theorem 2.8 does not apply.
- Example?

# Optimality of extreme points

- On the other hand, any linear programming problem can be transformed into an equivalent problem in standard form to which Theorem 2.8 does apply.
- This establishes the following corollary.

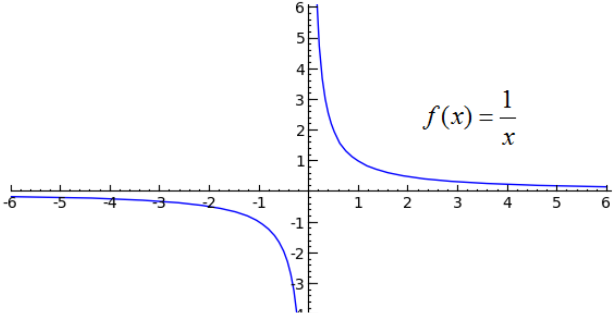
**Corollary 2.3**
Consider the linear programming problem of minimizing $c'x$ over a nonempty polyhedron. Then, either the optimal cost is equal to $-\infty$ or there exists an optimal solution.

# Optimality of extreme points

- The result in Corollary 2.3 should be contrasted with what may happen with a nonlinear cost function.
- For example, in the problem

$$\begin{aligned} \text{minimize} \quad & 1/x \\ \text{subject to} \quad & x \geq 1, \end{aligned}$$

the optimal cost is not $-\infty$, but an optimal solution does not exist.



# Why is Linear Programming special?

[2/3 spoilers understood...]

Property 1 [Corollary 2.3]:
Any linear programming problem is either infeasible, or it is unbounded, or it has an optimal solution.

Property 2:
In a linear programming problem each local optimal solution is also a global optimal solution.

Property 3 [Theorem 2.7]:
If a linear programming problem has an optimal solution and has at least one extreme point, then there exists an optimal solution that is an extreme point.

# 2.7 Representation of polytopes

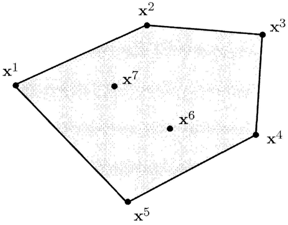
# Representation of polytopes

**Definition 2.5**

Let $x^1, \dots, x^k \in \mathbb{R}^n$ and $\lambda_1, \dots, \lambda_k \in \mathbb{R}$ be such that $\lambda_i \geq 0$ for $i = 1, \dots, k$ and $\sum_{i=1}^k \lambda_i = 1$.

(a) The vector $\sum_{i=1}^k \lambda_i x^i$ is said to be a convex combination of $x^1, \dots, x^k$.

(b) The convex hull of $x^1, \dots, x^k$ is the set of all possible convex combinations of $x^1, \dots, x^k$.



# Representation of polytopes

- So far, we have been representing polyhedra in terms of their defining inequalities.
- The next theorem provides an alternative:

**Theorem 2.9**
A nonempty polytope is the convex hull of its extreme points.

We will not prove this.

# Representation of polytopes

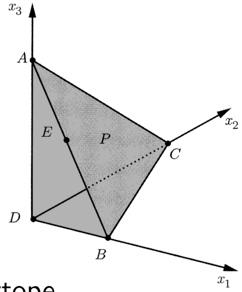
- So far, we have been representing polyhedra in terms of their defining inequalities.
- The next theorem provides an alternative:

**Theorem 2.9**
A nonempty polytope is the convex hull of its extreme points.

We will not prove this.

- There is a converse to Theorem 2.9 asserting that the convex hull of a finite number of points is a polyhedron.
- This result is proved in the next section.

# Example 2.6



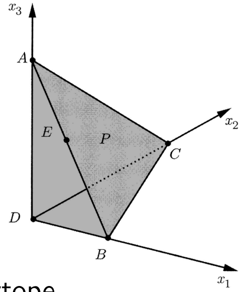

- Consider the polytope

$$P = \{(x_1, x_2, x_3) \mid x_1 + x_2 + x_3 \leq 1,\ x_1, x_2, x_3 \geq 0\}.$$

- It has four extreme points:

$$x^A = (0,0,1),\ x^B = (1,0,0),\ x^C = (0,1,0),\ x^D = (0,0,0).$$

# Example 2.6




- Consider the polytope

$$P = \{(x_1, x_2, x_3) \mid x_1 + x_2 + x_3 \leq 1,\ x_1, x_2, x_3 \geq 0\}.$$

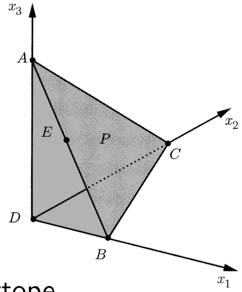
- It has four extreme points:

$$x^A = (0,0,1),\ x^B = (1,0,0),\ x^C = (0,1,0),\ x^D = (0,0,0).$$

- The vector $x = (1/3, 1/3, 1/4)$ belongs to $P$.
- It can be represented as

$$x = \frac{1}{4}x^A + \frac{1}{3}x^B + \frac{1}{3}x^C + \frac{1}{12}x^D.$$

# Example 2.6




- Consider the polytope

$$P = \{(x_1, x_2, x_3) \mid x_1 + x_2 + x_3 \leq 1,\ x_1, x_2, x_3 \geq 0\}.$$

- It has four extreme points:

$$x^A = (0, 0, 1),\ x^B = (1, 0, 0),\ x^C = (0, 1, 0),\ x^D = (0, 0, 0).$$

- Viceversa, consider the vector

$$\frac{1}{3}x^A + \frac{1}{6}x^B + \frac{1}{6}x^C + \frac{1}{3}x^D = \left(\frac{1}{6}, \frac{1}{6}, \frac{1}{3}\right).$$

- It can be checked that it belongs to $P$.

# 2.8 Projections of polyhedra: Fourier-Motzkin elimination

# Projections of polyhedra: Fourier-Motzkin elimination

- We present perhaps the oldest method for solving linear programming problems.
- This method is not practical because it requires a very large number of steps.
- However, it has some interesting theoretical corollaries.

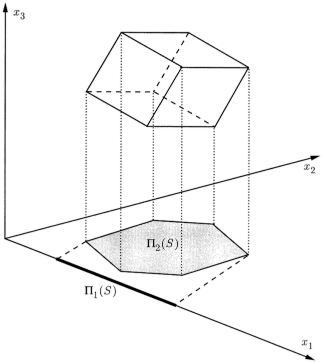
# Projections of polyhedra: Fourier-Motzkin elimination

- The key to this method is the concept of a projection, defined as follows:
- If $x = (x_1, \ldots, x_n)$ is a vector in $\mathbb{R}^n$ and $k \leq n$, the projection mapping

$$\pi_k : \mathbb{R}^n \mapsto \mathbb{R}^k$$

projects $x$ onto its first $k$ coordinates:

$$\pi_k(x) = \pi_k(x_1, \ldots, x_n) = (x_1, \ldots, x_k).$$

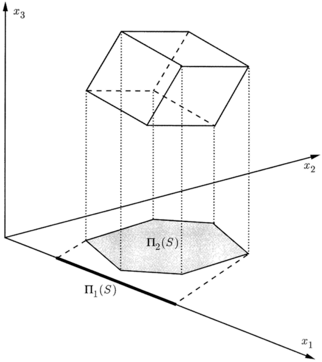


# Projections of polyhedra: Fourier-Motzkin elimination

- We also define the projection $\Pi_k(S)$ of a set $S \subseteq \mathbb{R}^n$ by letting

$$\Pi_k(S) = \{\pi_k(x) \mid x \in S\}.$$

- Note that $S$ is nonempty if and only if $\Pi_k(S)$ is nonempty.




- An equivalent definition is

$$\Pi_k(S) = \{(x_1, \dots, x_k) \mid \exists x_{k+1}, \dots, x_n \text{ s.t. } (x_1, \dots, x_n) \in S\}.$$

# Projections of polyhedra: Fourier-Motzkin elimination

- Consider the problem of deciding whether a given polyhedron $P \subseteq \mathbb{R}^n$ is nonempty.
- Suppose we can construct the set
$$\Pi_{n-1}(P) \subseteq \mathbb{R}^{n-1}.$$
- We can then consider the equivalent problem of deciding whether $\Pi_{n-1}(P)$ is nonempty.
- If we keep eliminating variables one by one, we eventually arrive at the set
$$\Pi_1(P) \subseteq \mathbb{R}.$$
- Its emptiness is easy to check. How?

# Projections of polyhedra: Fourier-Motzkin elimination

- The main disadvantage of this method is that while each step reduces the dimension by one, a large number of constraints is usually added.
- The number of constraints can increase exponentially with the problem dimension (see Exercise 2.20).

# Projections of polyhedra: Fourier-Motzkin elimination

We now describe the Fourier-Motzkin elimination method.

- We are given a polyhedron $P$:

$$P = \left\{ x \in \mathbb{R}^n \mid \sum_{j=1}^{n} a_{ij} x_j \geq b_i, \quad i = 1, \ldots, m \right\}.$$

- We wish to eliminate $x_n$ and construct the projection $\Pi_{n-1}(P)$.

# Projections of polyhedra: Fourier-Motzkin elimination

**Elimination algorithm**

1. Rewrite each constraint $\sum_{j=1}^{n} a_{ij} x_j \geq b_i$ in the form

$$a_{in} x_n \geq -\sum_{j=1}^{n-1} a_{ij} x_j + b_i, \qquad i = 1, \dots, m.$$

- If $a_{in} \neq 0$, divide both sides by $a_{in}$.
- By letting $\bar{x} = (x_1, \dots, x_{n-1})$, we obtain an equivalent representation of $P$:

$$\begin{aligned} x_n &\geq d_i + f_i' \bar{x} && \text{if } a_{in} > 0, \\ d_j + f_j' \bar{x} &\geq x_n && \text{if } a_{jn} < 0, \\ 0 &\geq d_k + f_k' \bar{x} && \text{if } a_{kn} = 0. \end{aligned}$$

Here, each $d_i, d_j, d_k \in \mathbb{R}$, and each $f_i, f_j, f_k \in \mathbb{R}^{n-1}$.

# Projections of polyhedra: Fourier-Motzkin elimination

**Elimination algorithm**

- By letting $\bar{x} = (x_1, \ldots, x_{n-1})$, we obtain an equivalent representation of $P$:

$$\begin{aligned} x_n &\geq d_i + f_i'\bar{x} && \text{if } a_{in} > 0, \\ d_j + f_j'\bar{x} &\geq x_n && \text{if } a_{jn} < 0, \\ 0 &\geq d_k + f_k'\bar{x} && \text{if } a_{kn} = 0. \end{aligned} \qquad (*)$$

  Here, each $d_i, d_j, d_k \in \mathbb{R}$, and each $f_i, f_j, f_k \in \mathbb{R}^{n-1}$.

2. Let $Q$ be the polyhedron in $\mathbb{R}^{n-1}$ defined by the constraints

$$\begin{aligned} d_j + f_j'\bar{x} &\geq d_i + f_i'\bar{x} && \text{if } a_{in} > 0 \text{ and } a_{jn} < 0, \\ 0 &\geq d_k + f_k'\bar{x} && \text{if } a_{kn} = 0. \end{aligned} \qquad (**)$$

# Example 2.7

Consider the polyhedron defined by the constraints

$$\begin{aligned} x_1 + x_2 &\geq 1 \\ x_1 + x_2 + 2x_3 &\geq 2 \\ 2x_1 + 3x_3 &\geq 3 \\ x_1 - 4x_3 &\geq 4 \\ -2x_1 + x_2 - x_3 &\geq 5. \end{aligned}$$

# Example 2.7

Consider the polyhedron defined by the constraints

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\begin{aligned}
x_1 + x_2 &\geq 1 \\
x_1 + x_2 + 2x_3 &\geq 2 \\
2x_1 + 3x_3 &\geq 3 \\
x_1 - 4x_3 &\geq 4 \\
-2x_1 + x_2 - x_3 &\geq 5.
\end{aligned}
$$

1. We rewrite these constraints in the form

$$
\begin{aligned}
0 &\geq 1 - x_1 - x_2 \\
x_3 &\geq 1 - x_1/2 - x_2/2 \\
x_3 &\geq 1 - 2x_1/3 \\
-1 + x_1/4 &\geq x_3 \\
-5 - 2x_1 + x_2 &\geq x_3.
\end{aligned}
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# Example 2.7

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-1 + x_1/4 &\geq x_3 \\
-5 - 2x_1 + x_2 &\geq x_3.
\end{aligned}
$$

# Example 2.7

2. Then, the set $Q$ is defined by the constraints

$$
\begin{aligned}
0 &\geq 1 - x_1 - x_2 \\
-1 + x_1/4 &\geq 1 - x_1/2 - x_2/2 \\
-1 + x_1/4 &\geq 1 - 2x_1/3 \\
-5 - 2x_1 + x_2 &\geq 1 - x_1/2 - x_2/2 \\
-5 - 2x_1 + x_2 &\geq 1 - 2x_1/3.
\end{aligned}
$$

1. We rewrite these constraints in the form

$$
\begin{aligned}
0 &\geq 1 - x_1 - x_2 \\
x_3 &\geq 1 - x_1/2 - x_2/2 \\
x_3 &\geq 1 - 2x_1/3 \\
-1 + x_1/4 &\geq x_3 \\
-5 - 2x_1 + x_2 &\geq x_3.
\end{aligned}
$$

# Projections of polyhedra: Fourier-Motzkin elimination

**Theorem 2.10**
The polyhedron $Q$ constructed by the elimination algorithm is equal to the projection $\Pi_{n-1}(P)$ of $P$.

Let's prove it!

# Projections of polyhedra: Fourier-Motzkin elimination

- Notice that for any vector $x = (x_1, \dots, x_n)$, we have

$$\pi_{n-2}(x) = (x_1, \dots, x_{n-2}) = \pi_{n-2}(\pi_{n-1}(x)).$$

- Accordingly, for any polyhedron $P$, we also have

$$\Pi_{n-2}(P) = \Pi_{n-2}(\Pi_{n-1}(P)).$$

- By generalizing this observation, we see that the set $\Pi_1(P)$ can be obtained by applying the elimination algorithm $n-1$ times.

# Projections of polyhedra: Fourier-Motzkin elimination

- Each application of the elimination algorithm can increase the number of constraints substantially.
- The polyhedron $\Pi_1(P)$ obtained may be described by a very large number of constraints.
- Of course, since $\Pi_1(P)$ is one-dimensional, almost all of these constraints will be redundant.
- This is of no help: in order to decide which ones are redundant, we must, in general, enumerate them.

# Projections of polyhedra: Fourier-Motzkin elimination

The elimination algorithm has an important theoretical consequence:

- The projection $\Pi_k(P)$ can be generated by repeated application of the elimination algorithm.
- The elimination algorithm always produces a polyhedron.

$\Rightarrow$ A projection $\Pi_k(P)$ of a polyhedron is also a polyhedron.

**Corollary 2.4**

Let $P \subset \mathbb{R}^{n+k}$ be a polyhedron. Then, the set

$$\{x \in \mathbb{R}^n \mid \text{there exists } y \in \mathbb{R}^k \text{ such that } (x, y) \in P\}$$

is also a polyhedron.

# Projections of polyhedra: Fourier-Motzkin elimination

**Corollary 2.6**
The convex hull of a finite number of vectors is a polyhedron.

Proof:

- The convex hull of $x^1, \ldots, x^k$ is the set

$$\{x \in \mathbb{R}^n \mid \text{there exists } \lambda \in \mathbb{R}^k \text{ such that } (x, \lambda) \in P\},$$

where $P = \left\{(x, \lambda) \mid x = \sum_{i=1}^{k} \lambda_i x^i,\ \sum_{i=1}^{k} \lambda_i = 1,\ \lambda \geq 0\right\}.$

- Corollary 2.4 implies that it is a polyhedron.

# Projections of polyhedra: Fourier-Motzkin elimination

The elimination algorithm can be used to solve linear programming problems.

- Consider the linear programming problem

$$\min\{c'x \mid Ax \geq b\}.$$

- Define a new variable $x_0$ and consider the set

$$\{(x_0, x) \mid Ax \geq b,\ x_0 = c'x\}.$$

- If we use the elimination algorithm $n$ times to eliminate the variables $x_1, \ldots, x_n$, we are left with the set

$$Q = \{x_0 \mid \text{there exists } x \text{ such that } Ax \geq b,\ x_0 = c'x\},$$

  and the optimal cost is equal to the smallest element of $Q$.
- An optimal solution $x$ can be recovered by backtracking (Exercise 2.21).