ISyE/Math/CS/Stat 525 – Linear Optimization
Spring 2020

Assignment 2

Due date: February 25 at 2pm.

Instructions and policy: This assignment contains two sections, one including mandatory exercises, and one with optional exercises, that serve for extra-practice.

Undergraduate students should handle in only the mandatory exercises that are marked with [U].
Graduate students should handle in only the mandatory exercises that are marked with [G].

The assignments should be submitted electronically in Canvas. Late submission policy: 10% of total points will be deducted per hour. IMPORTANT: Plan on submitting well before the deadline. If a technical problem occurs, and you cannot resolve it by the deadline, send an email to the TA before the deadline and attach your solution.

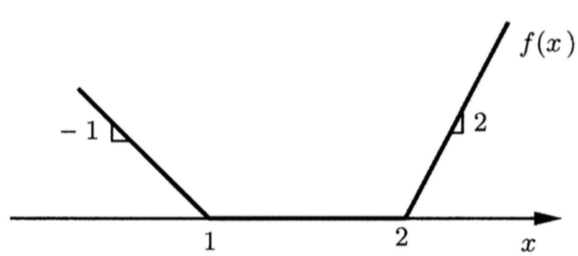
Students are strongly encouraged to work in groups of two on homework assignments. To find a partner you can post on the "Discussions" section in Canvas. Only one file should be submitted for both group members. In order to submit the assignment for your group please follow these steps in Canvas: Step 1. Click on the "People" tab, then on "Assignment", and join one of the available groups for the assignment; Step 2. When also your partner has joined the same group, one of the two can submit the assignment by clicking on the "Assignments" tab, then on the assignment to be submitted, and finally on "Submit assignment". The submission will count for everyone in your group.

Groups must work independently of each other, may not share answers with each other, and solutions must not be copied from the internet or other sources. If improper collaboration is detected, *all groups* involved will automatically receive a 0. Students must properly give credit to any outside resources they use (such as books, papers, etc.). In doing these exercises, you must justify all of your answers and cite every result that you use. You are not allowed to share any content of this assignment.

Mandatory exercises

Exercise 1 [U] .... ***4 points***

Consider the problem of minimizing a cost function of the form $c'x + f(d'x)$, subject to the linear constraints $Ax \geq b$. Here, d is a given vector and the function $f : \mathbb{R} \mapsto \mathbb{R}$ is described in the picture below.



Provide a linear programming formulation of this problem.

Exercise 2 [U][G] .... ***6 points***

Consider the following optimization problems involving absolute values. For each one, give an equivalent linear program using similar ideas to the ones seen in class, and prove the equivalence.

(a) (2 points)

$$\begin{aligned} \min \quad & -2x_1 + 3|x_2 - 10| \\ \text{s.t.} \quad & |x_1 + 2| + |x_2| \le 5 \end{aligned}$$

(b) (2 points)

$$\begin{aligned} \max \quad & x_1 - |x_2 + 3| \\ \text{s.t.} \quad & |x_1| + x_2 \le 3 \\ & x_2 \ge 0 \end{aligned}$$

(c) (2 points)

$$\begin{aligned} \min \quad & |x_1| - |x_2 - 1| \\ \text{s.t.} \quad & 3x_1 + 2x_2 \ge 1 \\ & x_2 \ge 1 \end{aligned}$$

Exercise 3 [U][G] .... ***5 points***

Consider a road divided into n segments that is illuminated by m lamps. Let p_j be the power of the jth lamp. The illumination I_i of the ith segment is assumed to be $\sum_{j=1}^{m} a_{ij}p_j$, where a_{ij} are known coefficients. Let I_i^* be the desired illumination of segment i.

We are interested in choosing the lamp powers p_j so that the illuminations I_i are close to the desired illuminations I_i^*. Provide a reasonable linear programming formulation of this problem. Note that the wording of the problem is loose and there is more than one possible formulation.

Exercise 4 .... ***8 points***

Consider an optimization problem (P) with absolute values in the following form:

$$\begin{aligned} \min \quad & c'x + d'y \\ \text{s.t.} \quad & Ax + By \le b \\ & y_i = |x_i| \qquad \forall\, i, \end{aligned}$$

and assume that all entries of B and d are nonnegative.

(a) (2 points) **[U][G]** Provide a linear programming reformulation of the above problem, using ideas similar to the ones discussed in class.

(b) (4 points) **[G]** Show that the original problem and the reformulation are equivalent.

(c) (2 points) **[U][G]** Provide an example to show that if B has negative entries, the problem may have a local minimum that is not a global minimum.

Exercise 5 [U][G] .... ***6 points***

Consider the polyhedron defined by the following system of linear inequalities:

$$\begin{array}{rrrrrrrrrrrr} - & x_1 & + & x_2 & + & x_3 & + & 2x_4 & - & 2x_5 & = & 1 \\ & 3x_1 & + & 2x_2 & + & 7x_3 & - & x_4 & + & 4x_5 & = & 7 \\ - & 2x_1 & + & 4x_2 & + & 6x_3 & + & 2x_4 & - & x_5 & = & 6 \\ & & & & & & & & & x & \ge & 0. \end{array}$$

Consider the three vectors $(1/2, 1, 1/2, 0, 0)$, $(1, 2, 0, 0, 0)$ and $(1, 0, 0, 1, 0)$. For each vector:

(a) (3 points) Say if the vector is a basic solution and, if so, specify if the basic solution is degenerate and determine all the bases associated to it.

(b) (3 points) Say if the vector is a vertex and, if so, determine an objective function that is uniquely minimized at the vertex.

Exercise 6 [U][G] .... ***5 points***

Let P be a bounded polyhedron in $\mathbb{R}^n$, let a be a vector in $\mathbb{R}^n$, and let b be some scalar. We define

$$Q = \{x \in P \mid a'x = b\}.$$

Show that every extreme point of Q is either an extreme point of P or a convex combination of two adjacent extreme points of P. *Hint*: Look at the proof of Theorem 2.6 $(b) \to (a)$ in the book. The idea of this proof has been explained in class.

Optional exercises

Exercise 7 .... ***0 points***

A mapping f is called *affine* if it is of the form $f(x) = Ax + b$, where A is a matrix and b is a vector. Let P and Q be polyhedra in $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively. We say that P and Q are *isomorphic* if there exist affine mappings $f : P \mapsto Q$ and $g : Q \mapsto P$ such that $g(f(x)) = x$ for all $x \in P$, and $f(g(y)) = y$ for all $y \in Q$. (Intuitively, isomorphic polyhedra have the same shape.)

(a) If P and Q are isomorphic, show that there exists a one-to-one correspondence between their extreme points. In particular, if f and g are as above, show that x is an extreme point of P if and only if $f(x)$ is an extreme point of Q.

(b) **(Introducing slack variables leads to an isomorphic polyhedron)** Let $P = \{x \in \mathbb{R}^n \mid Ax \geq b,\ x \geq 0\}$, where A is a matrix of dimensions $k \times n$. Let $Q = \{(x, z) \in \mathbb{R}^{n+k} \mid Ax - z = b,\ x \geq 0,\ z \geq 0\}$. Show that P and Q are isomorphic.

Exercise 8 .... ***0 points***

Consider the standard form polyhedron $P = \{x \in \mathbb{R}^n \mid Ax = b,\ x \geq 0\}$. Suppose that the matrix A, of dimension $m \times n$, has linearly independent rows, and that all basic feasible solutions are nondegenerate. Let x be a vector in P that has exactly m positive components.

(a) Show that x is a basic feasible solution. *Hint*: Look at the proof of Theorem 2.6 $(b) \to (a)$ in the book. The idea of the proof has been explained in class.

(b) Show that the result of part (a) is false if the nondegeneracy assumption is removed.